

$$3) \quad T(1,1) = (1,2) \quad T(0,1) = (2,1) \quad T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$\text{Given Basis} = \{(1,1), (0,1)\}$$

$$\begin{aligned} (x,y) &= \alpha_1 (1,1) + \alpha_2 (0,1) \\ &= (\alpha_1, \alpha_1 + \alpha_2) \\ \alpha_1 &= x, \quad \alpha_1 + \alpha_2 = y \Rightarrow \alpha_2 = y - x \\ \rightarrow T(x,y) &= \alpha_1 T(1,1) + \alpha_2 T(0,1) \end{aligned}$$

$$\begin{aligned} &= x(1,2) + (y-x)(2,1) \\ &= x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (y-x) \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &\rightarrow (x + 2y - 2x, 2x + y - x) \end{aligned}$$

$$\begin{aligned} T(x,y) &= (-x + 2y, x + y) \\ &= \begin{bmatrix} x + 2y - 2x \\ 2x + y - x \end{bmatrix} = \begin{bmatrix} -x + 2y \\ x + y \end{bmatrix} \end{aligned}$$

$$2) \quad T(1,1,0) = (1,2,3), \quad T(0,1,1) = (1,0,1) \quad T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$\{(1,1,0), (0,1,1)\}$$

$$\begin{aligned} (2,3,1) &= \alpha_1 (1,1,0) + \alpha_2 (0,1,1) \\ &= (\alpha_1, \alpha_1 + \alpha_2, \alpha_2) \\ \alpha_1 &= 2, \quad \alpha_1 + \alpha_2 = 3, \quad \alpha_2 = 1 \\ \rightarrow T(2,3,1) &= \alpha_1 T(1,1,0) + \alpha_2 T(0,1,1) \\ &= 2(1,2,3) + 1(1,0,1) \\ &= (3,4,7) \end{aligned}$$

$$\begin{aligned} (2,0,0) &= (\alpha_1, \alpha_1 + \alpha_2, \alpha_2) \\ \alpha_1 &= 2, \quad \alpha_1 + \alpha_2 = 0, \quad \alpha_2 = -2 \end{aligned}$$

$$\chi \geq 2, d_1 + d_2 \geq 0, a_2 = 0$$

$T(2,0,0)$  can't be determined.

$$\dim V = \dim N(T) + \dim \mathcal{R}(T)$$

$$\begin{array}{ccc} 3 & \boxed{\begin{array}{c} 0 \\ 1 \end{array}} & 3 \\ 3 & & 2 \end{array}$$

i) d)  $T(x, y, z) = (xy, yz, zx)$

$$T(x+y) = T(x) + T(y)$$

$$x = (1, 0, 0), y = (0, 1, 0)$$

$$T[(1, 0, 0) + (0, 1, 0)] = T(1, 0, 0) + T(0, 1, 0)$$

$$T(1, 1, 0) = (0, 0, 0) + (0, 0, 0)$$

$$(1, 0, 0) = (0, 0, 0) \quad \neq$$

b)  $T(x, y) = (2x + y, x - 2y + 1)$

$$T(0, 0) = (0, 1)$$

$$V \quad U^3$$

f)  $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a+b, a-b, a) \quad T: M_{2 \times 2} \rightarrow \mathbb{R}^3$

$$x = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \quad y = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$T(\alpha x + \beta y) = T \left[ \alpha \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \beta \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \right]$$

$$= T \begin{bmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 \\ \alpha c_1 + \beta c_2 & \alpha d_1 + \beta d_2 \end{bmatrix}$$

$$= (\alpha a_1 + \beta a_2 + \alpha b_1 + \beta b_2, \alpha a_1 + \beta a_2 - \alpha b_1 - \beta b_2, \alpha a_1 + \beta a_2)$$

$$\alpha T(x) + \beta T(y) = \alpha T \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \beta T \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$= \alpha (a_1 + b_1, a_1 - b_1, a_1) + \beta (a_2 + b_2, a_2 - b_2, a_2)$$

$$= (\alpha a_1 + \alpha b_1 + \beta a_2 + \beta b_2, \alpha a_1 - \alpha b_1 + \beta a_2 - \beta b_2, \alpha a_1 + \beta a_2)$$

$$N(T) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (0, 0, 0) \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{aligned} (a+b, a-b, a) &= (0, 0, 0) \\ a+b &= 0, a-b &= 0, a &= 0 \end{aligned} \right.$$

$$= \left\{ \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \right\} \quad a=0, b=0$$

$$\dim N(T) = 2$$

$$(0, 0, (c, d)) = c(0, 0, 1) + d(0, 0, 1)$$

$$\text{Basis of } N(T) = \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$R(T)$$

$$\text{Basis of } M_{2 \times 2} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = (1, 1, 1) \quad T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = (1, -1, 0)$$

$$T \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = (0, 0, 0) \quad T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = (0, 0, 0)$$

$$\text{Span } R(T) = \{ (1, 1, 1), (1, -1, 0), (0, 0, 0) \}$$

$$= \{ (1, 1, 1), (1, -1, 0) \} \rightarrow \text{L.I.}$$

$$= \text{Basis of } R(T) \quad \dim R(T) = 2$$

$$5) \quad T(x, y, z) = (3x + y, x + z, x - z)$$

$$T(1, 0, 0) = (3, 1, 1)$$

$$T(0, 1, 0) = (1, 0, 0)$$

$$T(1,0,0) = (1,0,0)$$

$$T(0,1,0) = (1,0,0)$$

$$T(0,0,1) = (0,1,-1)$$

$$\text{Corresponding Matrix} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Note: Find  $A^4$

eigenvalue value  $\rightarrow$  eigen vector  $\rightarrow P \rightarrow P^{-1}$

$$P^{-1} A P = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P^{-1} A^n P = D^n$$

$$P^{-1} A^4 P = D^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

$$A^4 = P D^4 P^{-1}$$